Ref. Unit - III

Linear Differential Equation

Date

$D = \frac{d}{dx}$

$\frac{d^2}{dx^2} = D^2$

Homogenous

$f(D)y = 0$

Non-h.

$f(D)y = r(x)$

General Solution

$y(x)$ = Complementary function

General Solution

$y(x) = CF +$ Particular integral

Roots of auxiliary equation	Complementary function
(I) Real and distinct equal	I
(a) α, α	(a) $(c_1 + x c_2) e^{\alpha x}$
(b) α, β	(b) $c_1 e^{\alpha x} + c_2 e^{\beta x}$
(c) α, α, α	(c) $(c_1 + x c_2 + x^2 c_3) e^{\alpha x}$
(d) α, α, β	(d) $(c_1 + x c_2) e^{\alpha x} + e^{\beta x}$
(II) Complex roots	
(a) $\alpha = a + i\beta$ $\beta = a - i\beta$	(a) $[c_1 \cos bx + c_2 \sin bx] e^{ax}$

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S2. $(D^2-3D-4)y = e^x + 6e^{5x}$

Solⁿ

AE - $D^2-3D-4=0.$

$\Rightarrow (D-4)(D+1)=0.$

$\Rightarrow D=4,-1.$

$y_c(x) = c_1e^{-x} + c_2e^{4x}$

$$y_p(x) = \frac{1}{f(D)}X(x)$$

$$= \frac{1}{D^2-3D-4}(e^x+6e^{5x})$$

$$= \frac{1}{D^2-3D-4}[e^x] + 6\frac{1}{D^2-3D-4}e^{5x}$$

$$= \frac{1}{1^2-3(1)-4}e^x + 6\frac{1}{5^2-3\times5-4}e^{5x}$$

$$= \frac{1}{-6}e^{-x} + \frac{6}{6}e^{5x}$$

$$= \frac{-e^x}{6} + e^{5x}.$$

Ex: $y'' - 6y' + 9y = 14e^{3x}$

Solⁿ:

AE - $D^2 - 6D + 9 = 0$

$(D-3)^2 = 0$

$\Rightarrow D = 3, 3$

$y_c(x) = (C_1 + xC_2)e^{3x}$

PI. or $y_p(x) = \dfrac{g(x)}{f(D)}$

$= \dfrac{14e^{3x}}{D^2 - 6D + 9}$ ⁄ $f(D) = 0$ [rule fail]

$y_p(x) = x\dfrac{14e^{3x}}{f'(D)} = x\dfrac{14e^{3x}}{2D-6}$ ⁄ $2D-6$

$2(3) - 6 = 0$.

rule fail

$y_p(x) = x^2 \dfrac{14e^{3x}}{f''(D)}$

$= x^2 \dfrac{\cancel{14}^{7} e^{3x}}{\cancel{2}} = 7x^2 e^{3x}$. Ans.

$\therefore$ General Solⁿ.

$y(x) = y_c(x) + y_p(x)$

$= (C_1 + xC_2)e^{3x} + 7x^2 e^{3x}$

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Non-homogeneous linear diff. equation

General Sol^n = $y_c + y_p$

CF — Particular Integral

$$a_0 y'' + a_1 y' + a_2 y = r(x)$$

Step1 SF

$$[a_0 D^2 + a_1 D + a_2] y = r(x)$$

Step2 AE.

$$\boxed{a_0 D^2 + a_1 D + a_2 = 0.}$$

$\downarrow$

Complementary function (y_c).

Imp.

$$\boxed{P.I = \frac{1}{f(D)} \cdot r(x)}$$

CASE I If $r(x) = e^{\alpha x}$. then $PI = \frac{1}{f(D)} \cdot e^{\alpha x}$

$$= \frac{1}{f(\alpha)} \cdot e^{\alpha x} \quad , f(\alpha) \neq 0$$

If $f(\alpha) = 0$, then $PI = x \frac{1}{f'(\alpha)} \cdot e^{\alpha x}$ $\quad f'(\alpha) \neq 0$.

If $f'(\alpha) = 0$, then $PI = x^2 \frac{1}{f''(\alpha)} e^{\alpha x}$, $f''(\alpha) \neq 0$

⋮

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Example Q1 $(D^2+5D+4)y = 18e^{2x}$

AE - $D^2+5D+4=0.$ $y(x) = y_c(x) + PI$

$(D+1)(D+4)=0$

$\Rightarrow D = -1, -4.$

$y_c = C_1e^{-x} + C_2e^{-4x}$

PI or y_p = $\frac{1}{f(D)} \cdot r(x) = \frac{1}{D^2+5D+4} 18e^{2x}$

$= \frac{1}{f(\alpha)} \cdot 18e^{2x} = \frac{1}{\alpha^2+5\alpha+4} 18e^{2x}$

$\boxed{\alpha = 2}$ $= \frac{18}{2^2+5\times 2+4} \cdot e^{2x}$

$= \frac{18}{18} e^{2x}$

$= e^{2x}$

General Solⁿ.

$\therefore y(x) = C_1e^{-x} + C_2e^{-4x} + e^{2x}.$ Ans

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$\boxed{\text{CASE II}}$

If $r(x) = \cos\alpha x$ or $\sin\alpha x$

$$y(x) \text{ or } PI = \frac{1}{f(D)} r(x)$$

$$\boxed{D^3 = D^2(D)}$$

Here, in this case $\boxed{D^2 = -\alpha^2}$

$$PI = \frac{1}{f(D^2 = -\alpha^2)} (\cos ax / \sin ax) \qquad f(D^2 = -\alpha^2) \neq 0$$

If $f(D^2 = -\alpha^2) = 0$, then,

$$PI = x \frac{1}{f'(D^2 = -\alpha^2)} \cos\alpha x \qquad f'(D^2 = -\alpha^2) \neq 0$$

If $f'(D^2 = -\alpha^2) = 0$, then

$$PI = x^2 \frac{1}{f''(D^2 = -\alpha^2)} \cos\alpha x$$

and so on.

Ex :- 1 $(D^2+1)y = \cos 2x$

AE - $D^2+1=0$ $\boxed{D^2 = -\alpha^2}$

$\Rightarrow D = \pm i$

$y_c = \text{C.F.} \; [C_1\cos x + C_2 \sin x]e^{0x} = [C_1 \cos x + C_2 \sin x]$

y_p or P.I $= \dfrac{r(x)}{f(D^2=-\alpha^2)}$

Here $\alpha = 2$

$\therefore D^2 = -2^2 = -4$

$= \dfrac{\cos 2x}{D^2+1}$

$= \dfrac{1}{-4+1}\cos 2x$

$= \dfrac{-1}{3}\cos 2x.$

$\therefore y(x) = y_c + y_p$

$= (C_1\cos x + C_2 \sin x) \cancel{+} - \frac{1}{3}\cos 2x.$ Ans

2 $(3D^2 - 7D + 2)y = \sin x + \cos x$

Solⁿ

AE - $3D^2 - 7D + 2 = 0$

$\Rightarrow (3D-1)(D-2) = 0$ Here

$\Rightarrow D = \frac{1}{3}, 2$

$y_c = C_1 e^{2x} + C_2 e^{\frac{1}{3}x}$

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$D^3 = D^2 \cdot D = -D$

$$\Rightarrow y_p(x) = \frac{1}{3D^2-7D+2}[\sin x + \cos x]$$

$$\Rightarrow y_p(x) = \frac{1}{3D^2-7D+2}[\sin x] + \frac{1}{(3D^2-7D+2)}\cos x$$

~~$\Rightarrow y_p(x)$~~ $D^2 = -1^2 = -1$ $D^2 = -1^2 = -1$

$$y_p(x) = \frac{1}{3(-1)-7D+2}\sin x + \frac{1}{3(-1)-7D+2}\cos x$$

$$y_p(x) = \frac{1}{-1-7D}\sin x + \frac{1}{-1-7D}\cos x$$

$$y_p(x) = \frac{(-1)}{(1+7D)}\sin x - \frac{1}{(1+7D)}\cos x$$

$$y_p(x) = \frac{(-1)(1-7D)}{(1+7D)(1-7D)}\sin x - \frac{(1-7D)}{(1+7D)(1-7D)}\cos x$$

$$y_p(x) = \frac{7D-1}{1^2-(7D)^2}(\sin x) - \frac{(1-7D)\cos x}{1^2-(7D)^2}$$

$$y_p(x) = \frac{(7D-1)(\sin x)}{50} - \frac{(1-7D)\cos x}{50}$$

$$y_p(x) = \left(\frac{7D\sin x - \sin x}{50}\right) - \left(\frac{\cos x - 7D\cos x}{50}\right)$$

$D = d/dx$

$$\Rightarrow y_p(x) = \frac{7\cos x - \sin x}{50} - \frac{[\cos x + 7\sin x]}{50}$$

$$\Rightarrow y_p(x) = \frac{6\cos x - 8\sin x}{50}.$$

$$y(x) = y_c + y_p$$

$$= C_1 e^{2x} + C_2 e^{1/3 x} + \frac{1}{50}[6\cos x - 8\sin x]. \quad \text{Ans}$$

Case III

If $r(x) = x^n$, $n > 0$, then

let $f(D) = a_0 D^2 + a_1 D + a_2$

$$y_p = \frac{x^n}{f(D)}$$

$$\Rightarrow y_p = \frac{x^n}{a_0 D^2 + a_1 D + a_2}$$

$$\Rightarrow y_p = \frac{1}{a_2\left[1 + \frac{a_0 D^2 + a_1 D}{a_2}\right]^1} x^n$$

$$\Rightarrow y_p = \frac{\overbrace{\left[1 + \frac{a_0 D^2 + a_1 D}{a_2}\right]^{-1}}}{a_2} x^n \quad \{\text{Expand this term}\}.$$

$$\boxed{\begin{aligned} &\underline{\text{Imp. formula.}} \\ &(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots \\ &(1-x)^{-1} = 1 + x + x^2 + x^4 + \dots \end{aligned}}$$

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$\Rightarrow$ Q find the particular integral of $(D^2+2D+1)y = x^3$

Sn.

$$y_P = \frac{1}{D^2+2D+1} \cdot x^3$$

$$\Rightarrow y_P = \frac{1}{[1+(D^2+2D)]^1} x^3$$

(D^4)

$$\Rightarrow y_P = [1+(D^2+2D)]^{-1} \cdot x^3$$

$$\Rightarrow y_P = [1-(D^2+2D)+(D^2+2D)^2-(D^2+2D)^3]x^3$$

48
24
72

$$\Rightarrow y_P = [1-2D-D^2+D^4+4D^2+4D^3+D^6 \ldots]x^3$$
$$8D^3+6D^4\ldots$$

$$\Rightarrow y_P = [x^3-2(3x^2)-6x+0+4(6x)+4(6)+8(6)$$

$$\Rightarrow y_P = [x^3-6x^2+18x+72].$$ Ans

Q $(D^2+25)y = 9x^3+4x^2$

Soln. AE. $D^2+25=0$

$\Rightarrow D = \pm 5i$

$y_c(x) = [C_1 \cos 5x + C_2 \sin 5x]$

y_p or $PI = \frac{1}{f(D)} r(x)$

$= \frac{1}{D^2+25}(9x^3+4x^2)$

$= \frac{1}{25\left(1+\frac{D^2}{25}\right)}(9x^3+4x^2)$

$= \frac{1}{25}\left[1+\frac{D^2}{25}\right]^{-1}(9x^3+4x^2)$

$= \frac{1}{25}\left[1-\frac{D^2}{25}+\left(\frac{D^2}{25}\right)^2+\dots\right](9x^3+4x^2)$

$= \frac{1}{25}\left[(9x^3+4x^2) - \frac{1}{25}[54x+8]\right]$

$= \frac{1}{25}\left[9x^3+4x^2 - \frac{54x}{25} - \frac{8}{25}\right]$

general soln $=$ $y_c + y_p$

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Case IV

If $r(x) = e^{\alpha x} h(x)$

$h(x) = \sin ax / \cos ax$

$= x^n$

$$y_p(x) = \frac{1}{f(D)} e^{\alpha x} h(x)$$

$$\Rightarrow y_p(x) = e^{\alpha x} \frac{1}{f(D+\alpha)} h(x)$$

Ex:- 1 $(D^2 - 4D + 5)y = 24e^{2x}\sin x$

Solm.

AE $= D^2 - 4D + 5 = 0$.

$\Rightarrow (D+1)(D-5) = 0$

$\Rightarrow D = -1, 5$

$\therefore y_c(x) = C_1 e^{-x} + C_2 e^{5x}$

$$y_p(x) = \frac{1}{(D^2 - 4D + 5)} (24e^{2x}\sin x)$$

$$y_p(x) = 24e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 5} (\sin x)$$

$$y_p(x) = 24e^{2x} \frac{1}{D^2 + 4 + \cancel{4D} - \cancel{4D} - 8 + 5} (\sin x)$$

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$$y_p(x) = 24 e^{2x} \times x \frac{1}{D^2+1} (\sin x)$$

$$\therefore \quad y_p(x) = e^{2x} x \frac{1}{2D} \sin x$$

$D^2 = -\alpha^2$
$= -1$

$\boxed{\because D^2+1=0}$

$$y_p(x) = \frac{e^{2x}}{2} \times 24 \, x \, \frac{1}{D} (\sin x)$$

$$= 12 e^{2x} x \int \sin x \, dx$$

$$= -12 e^{2x} x \cos x . \quad \text{Ans.}$$

$$\boxed{\begin{aligned} & D \frac{1}{D} \sin x \\ & \frac{1}{D} \times \frac{D}{D} \sin x \\ & \frac{D}{D^2} \sin x \\ & \frac{D (\sin x)}{-1} \\ & = -\cos x . \end{aligned}}$$

Rules: $f(D) y = x \, h(x)$

$$I = \frac{1}{f(D)} x . h(x) = x . \frac{1}{f(D)} h(x) + \frac{d}{dD}\left(\frac{1}{f(D)}\right) h(x)$$

$$= x . \frac{h(x)}{f(D)} - \frac{f'(D)}{[f(D)]^2} h(x).$$

Ex:- find the particular integral of $(D^2+4D+3)y = x \sin 2x$

$(\alpha = 2)$

$$PI = x . \frac{1}{D^2+4D+3} (\sin 2x) - \frac{2D+4}{(D^2+4D+3)^2} \sin 2x.$$

~~$PI = x . \frac{1}{-4+4D+3} (\sin 2x) - \frac{2D+4}{D^4+16D^2+9+}$~~

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$$PI = x \cdot \frac{1}{(-4+4D+3)} \sin 2x - \frac{2D+4}{[-4+4D+3]^2} \sin 2x$$

$$PI = x \cdot \frac{1}{(4D-1)} \sin 2x - \frac{2D+4}{4D-1} (\sin 2x)$$

$$PI = x \cdot \frac{4D+1}{16D^2-1} \sin x - \frac{(4D+1)(2D+4)}{16D^2-1} (\sin 2x)$$

$$\cancel{PI = x \cdot} \ \frac{\quad}{\quad}$$

$$PI = x \cdot \frac{[4\cos x + \sin x]}{-65} - \frac{[8D^2+18D+4] \sin 2x}{(-65)}$$

$$PI = -x \cdot \frac{[4\cos x + \sin x]}{65} + \frac{[-32\sin 2x + 36\cos 2x + 4\sin 2x]}{-65}$$

Q. $(D^2+9)y = xe^{2x} \cos x$.

Euler-Cauchy Equation

The differential equation of

$$a_0 x^n y^{(n)} + a_1 x^{(n-1)} y^{(n-1)} + a_2 x^{(n-2)} y^{(n-2)} + \dots + a_n y = r(x)$$

this type is called euler cauchy equation.

Imp: To solve the euler cauchy equation, we have to first convert this equation into diff. equation with constant coefficient,

Put $x = e^t$ i.e $t = \log x$

$$\frac{d}{dx} = \frac{1}{x}\frac{d}{dt} \Rightarrow x\frac{d}{dx} = \frac{d}{dt}$$

$$\Rightarrow xD = \theta$$

Also. $$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right)$$

$$= \frac{d}{dx}\left[\frac{1}{x}\frac{d}{dt}\right] = \frac{-1}{x^2}\frac{d}{dt} + \frac{1}{x}\cdot\frac{d}{dx}\left[\frac{d}{dt}\right]$$

$$= \frac{-1}{x^2}\frac{d}{dt} + \frac{1}{x}\cdot\frac{d}{dt}\left[\frac{d}{dt}\right]\frac{dt}{dx}$$

$$= \frac{-1}{x^2}\frac{d}{dt} + \frac{1}{x^2}\frac{d^2}{dt^2}$$

$$\frac{d^2}{dx^2} \Rightarrow \frac{1}{x^2}\left[\theta^2 - \theta\right]$$

$$x^2\frac{d^2}{dx^2} = \theta(\theta - 1)$$

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$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^3D^3 = \theta(\theta-1)(\theta-2)$

Q Find the general solution of the equation

$$2x^2y'' + 3xy' - 3y = x^3 .$$

Soln. SF $\quad 2x^2D^2 + 3xD - 3y = x^3$

To transform, put $x = e^t$ and $t = \log x$ and

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$

$$\rightarrow [2\theta(\theta-1) + 3\theta - 3]y = e^{3t}$$

$$\Rightarrow [2\theta^2 + \theta - 3]y = e^{3t}$$

AE $\quad 2\theta^2 + \theta - 3 = 0.$

$\Rightarrow \theta = 1, -3/2$

$$y_c(x) = c_1e^t + c_2e^{-\frac{3}{2}t}$$

$$y_P = \frac{1}{2\theta^2+\theta-3} e^{3t} \qquad \theta = 3$$

$$= \frac{1}{2(9)+(3)-3} e^{3t}$$

$$= \frac{1}{18} e^{3t}$$

$\therefore$ General solⁿ

$$y(t) = C_1 e^{t} + C_2 e^{-\frac{3}{2}t} + \frac{1}{18} e^{3t}$$

$$\boxed{y(x) = C_1 x + C_2 \cancel{x}^{-3/2} + \frac{1}{18} x^3}$$ Ans

Q find the general solution of $x^2y'' + 2xy' = \cos \ln(x)$

SF $\quad x^2D^2 + 2xD = \cos(\ln x)$

Put $x = e^t$ and $\ln x = t$ and $xD = \theta$

SF $\quad [\theta(\theta-1) + 2\theta]y = \cos t$

$\Rightarrow \quad [\theta^2 + \theta]y = \cos t$

AE $\quad \theta^2 + \theta = 0$

$\Rightarrow \theta(\theta+1) = 0$

$\Rightarrow \theta = 0, -1$

$y_c(t) = C_1 e^{0t} + C_2 e^{-t} = C_1 + C_2 e^{-t}$

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$\theta^2 = -1$

$\theta^2 + 1 = 0$

$\Rightarrow$ rule fails

$$y_P = \frac{1}{\theta^2 + \theta}(\cos t)$$

$$\therefore \quad y_P = \frac{1}{-1 + \theta}\cos t$$

$$\Rightarrow \quad y_P = \frac{\theta + 1}{(\theta - 1)(\theta + 1)}\cos t$$

$$\Rightarrow \quad y_P = \frac{[-\sin t + \cos t]}{\theta^2 - 1}$$

$$\Rightarrow y_P = \frac{[\cos t - \sin t]}{-2}$$

$\therefore$ Gen. Sol^n:

$$y(t) = c_1 + c_2 e^{-t} - \frac{\cos t}{2} + \frac{1}{2}\sin t$$

$$y(x) = c_1 + \frac{c_2}{x} - \frac{1}{2}\cos(\ln x) + \frac{1}{2}\sin(\ln x)$$

QQ. find the particular solution of the given differential equation.

Q

$x^2y'' - xy' + 2y = 6$, $y(1) = 1$ $y'(1) = 2$

SF

$(x^2D^2 - xD + 2)y = 6$

Put $x = e^t$, $t = \ln x$, $xD = \theta$, $x^2D^2 = \theta(\theta-1)$

$\Rightarrow [\theta(\theta-1) - \theta + 2]y = 6.$

$\Rightarrow [\theta^2 - 2\theta + 2]y = 6.$

AE $\theta^2 - 2\theta + 2 = 0.$

$$\theta = \frac{2 \pm \sqrt{4-8}}{2}$$

$\theta = 1 \pm i$

$y_c(t) = e^x[c_1\cos x + c_2\sin x]$

$$y_p(t) = 6\frac{1}{\theta^2 - 2\theta + 2}[e^{0t}]$$

$$= 6\frac{1}{2} \times e^{0t} = 3.$$

General Solution

$y(t) = e^t[c_1\cos t + c_2\sin t] + 3$

or $y(x) = x[c_1\cos(\ln x) + c_2\sin(\ln x)] + 3$

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Now, we know,

$y(1) = 1$

$\Rightarrow \quad 1 = 1[c_1 + 0] + 3 \quad \Rightarrow c_1 = -3$

and $y'(1) = 2$

$\Rightarrow \quad y'(x) = [c_1 \cos(\ln x) + c_2 \sin(\ln x)] + x\left[-c_1 \frac{\sin(\ln x)}{x} + \frac{c_2 \cos(\ln x)}{x}\right] + 0$

$\Rightarrow \quad y'(1) = 2$

$\Rightarrow \quad 2 = [c_1 + 0] + [-c_1 \times 0 + c_2 \times 1]$

$\Rightarrow \quad 2 = c_1 + c_2$

$\Rightarrow \quad 2 = -3 + c_2$

$\Rightarrow \quad \boxed{c_2 = 5}$

$y(x) = x[-3\cos(\ln x) + 5\sin(\ln x)]$. Ans

Method of undetermined coefficient

$$f(D)y = r(x)$$

$r(x)$	y_p by trial method	
$e^{\alpha x}$	$y_p = Ae^{\alpha x}$	α should not satisfy $f(D)$ or $e^{\alpha x}$ should not be present in C.F
	$y_p = xAe^{\alpha x}$	If $e^{\alpha x}$ if not present in C.F
	$y_p = x^2 A e^{\alpha x}$	If $xe^{\alpha x}$ is also present in C.F

Q find the general solution of the equation $y'' + 4y' + 4y = 12e^{-2x}$ by method of undetermined coefficient

Soln:- The given equation is

$$y'' + 4y' + 4y = 12e^{-2x}$$

SF $\quad [D^2 + 4D + 4]y = 12e^{-2x}$

AE $\Rightarrow D^2 + 4D + 4 = 0$

$\Rightarrow (D+2)^2 = 0.$

$\Rightarrow D = -2, -2.$

$\therefore y_c(x) = (c_1 + xc_2)e^{-2x}$

$= c_1 e^{-2x} + xc_2 e^{-2x}$

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The trial PI is

$$\cancel{A} \quad y_p = Ax^2e^{-2x}$$

$$y_p' = A[x^2e^{-2x} \times (-2) + 2x(e^{-2x})]$$

$$= A[2x - 2x^2]e^{-2x}$$

$$\Rightarrow y_p'' = A[(2x-2x^2)e^{-2x}(-2) + (2-4x)e^{-2x}]$$

$$\Rightarrow \quad = A[-4x + 4x^2 + 2 - 4x]e^{-2x}$$

$$y_p'' = A[2 - 8x + 4x^2]e^{-2x}$$

Since y_p is a solution of given differential equation

$\therefore$ y_p should satisfy the given differential equation

i.e, $y_p'' + 4y_p' + 4y_p = 12e^{-2x}$

$$A[2-8x+4x^2]e^{-2x} + 4[A(2x-2x^2]e^{-2x} + 4Ax^2e^{-2x} = 12e^{-2x}$$

$$A\cancel{e^{-2x}}[2 - \cancel{8x} + \cancel{4x^2} + \cancel{8x} - \cancel{8x^2} + \cancel{4x^2}] = 12\cancel{e^{-2x}}$$

$$\cancel{2A} \quad 2A = 12$$

$$\Rightarrow \boxed{A = 6}$$

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$y_p = 6x^2e^{-2x}$

$y(x) = (C_1 + xC_2)e^{-2x} + 6x^2e^{-2x}$.

$r(x)$		
$\cos\alpha x$ or $\sin\alpha x$	$y_p = A(\cos\alpha x + B\sin\alpha x)$ $y_p = x[A\cos\alpha x + B\sin\alpha x]$ $y_p = x^2[A\cos\alpha x + B\sin\alpha x]$	If $A\cos\alpha + B\sin\alpha$ should not be present in the CF not present in CF
$e^{ax}\sin bx$ or $e^{ax}\cos bx$	$y_p = e^{ax}[A\cos\alpha x + B\sin\alpha x]$ $y_p = e^{ax}x[A\cos\alpha x + B\sin\alpha x]$ $y_p = e^{ax}x^2[A\cos\alpha x + B\sin\alpha x]$	Same ~~then~~ applied here
x^n	$y_p = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots$ $[1 + D + D^2 + D^3 + \dots]x^n$ $x^n + nx^{n-1} + \dots$	

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Q. $y'' - 3y' - 10y = 1 + x^2$ ——— (1)

$\Rightarrow (D^2 - 3D - 10)y = 1 + x^2$

AE $D^2 - 3D - 10 = 0$

$(D-5)(D+2) = 0$

$\Rightarrow D = 5, -2.$

$Y_c = C_1 e^{-2x} + C_2 e^{5x}$

By hit and trial method

$y_P = Ax^2 + Bx + C$

$y_P' = 2Ax + B$

$y_P'' = 2A$

Now substitute, in Eq (1)

$\Rightarrow (2A)' - 3[2Ax + B] - 10[Ax^2 + Bx + C] = 1 + x^2$

$\Rightarrow 2A - 6Ax - 3B - 10Ax^2 - 10Bx - 10C = 1 + x^2.$

$\Rightarrow -10Ax^2 - x(10B + 6A) + (2A - 3B - 10C) = 1 + x^2$

Equating like terms,

$$-10A = 1 \quad \text{——— (1)}$$

$$2A - 3B - 10C = 1 \quad \text{——— (3)}$$

$$-6A - 10B = 0 \quad \text{——— (4)}$$

$A = -1/10$ from Eq (1).

Substituting $A = -1/10$ in Eq (4).

$$-6 \times \frac{-1}{10} - 10B = 0 \Rightarrow \frac{3}{5} = 10B \Rightarrow B = \frac{3}{50}$$

Substitute $A = -1/10$ and $B = 3/50$ in Eq (3)

$$\Rightarrow \quad 2 \times \frac{-1}{10} - 3 \times \frac{3}{50} - 10C = 1$$

$$\Rightarrow \quad \frac{-10}{50} - \frac{9}{50} - 10C = 1$$

$$\Rightarrow \quad -\frac{19}{50} - 10C = 1$$

$$\Rightarrow \quad -10C = 1 + \frac{19}{50}$$

$$\Rightarrow \quad -C = \frac{69}{500} \Rightarrow C = -\frac{69}{500}.$$

$$\therefore y_P = -\frac{1}{10}x^2 + \frac{3}{50}x - \frac{69}{500}$$

$$\boxed{\text{General Sol}^n = y_c + y_P}$$

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Q $y'' + 3y' + 2y = \cancel{2}\cos x + \sin x$ — (1)

SE $(D^2 + 3D + 2)y = \cos x + \sin x$

AE $D^2 + 2D + D + 2 = 0$

$(D+2)(D+1) = 0$

$\Rightarrow \boxed{D = -1, -2}$

$y_c(x) = C_1 e^{-x} + C_2 e^{2x}$

Let the trial solⁿ is

$y_p(x) = A\cos x + B\sin x$

$y'_p(x) = -A\sin x + B\cos x$

$y''_p(x) = -A\cos x - B\sin x$

Substitute the value in eq (1).

$-A\cos x - B\sin x + 3[-A\sin x + B\cos x] + 2[A\cos x + B\sin x] = \cancel{6}\cos x + \sin x$

$\Rightarrow [-A + 3B + 2A]\cos x + [-B - 3A + 2B]\sin x = \cos x + \sin x$

$\Rightarrow (A + 3B)\cos x + (-3A + B)\sin x = \cos x + \sin x$

$A + 3B = 1$ — (2) and $-3A + B = 1$ — (3).

Solving (2) and (3)

we get $A = -\frac{1}{5}$ and $B = \frac{2}{5}$

$$\boxed{y_p = -\frac{1}{5}\cos x + \frac{2}{5}\sin x}$$

General Solⁿ = $Y_c + Y_p$

Q $y'' - 6y' + 13y = 6e^{3x}\sin x\cos x$ —— (1)

$\Rightarrow y'' - 6y' + 13y = 3e^{3x}[2\sin x\cos x]$

$\Rightarrow y'' - 6y' + 13y = 3e^{3x}\sin 2x$

$\Rightarrow$ SF $(D^2 - 6D + 13)y = 3e^{3x}\sin 2x$

AE $D^2 - 6D + 13 = 0$

$D = 3 \pm 2i$

$y_c(x) = e^{3x}[c_1\cos 2x + c_2\sin 2x]$

By hit and trial method

$y_p = xe^{3x}[A\cos 2x + B\sin 2x]$

Now, find y_p' and y_p'' and Substitute in Eq (1)

& find value of A and B.

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Q $y''' - 2y'' + 4y' - 8y = 8(x^2 + \cos 2x)$ —— (1)

SF $\Rightarrow (D^3 - 2D^2 + 4D - 8)y = 8(x^2 + \cos 2x)$

AE $\Rightarrow D^3 - 2D^2 + 4D - 8 = 0$

$\Rightarrow (D^2+4)(D-2) = 0$

$\Rightarrow D = 2, \pm 2i.$

$\therefore y_c(x) = c_1 e^{2x} + [c_2 \cos 2x + c_3 \sin 2x]$

By trial method,

$$y_P = (Ax^2 + Bx + C) + x[D\cos 2x + E\sin 2x]$$

Now, find y_P', y_P'' and y_P''' and subsitute in equation (1) to get value A and B.

~~fun~~

Imp METHOD OF VARIATION OF PARAMETER

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$$

$$y_c(x) = C_1\, y_1(x) + C_2\, y_2(x)$$

$$y_p(x) = A(x)\, y_1(x) + B(x)\, y_2(x)$$

~~where~~, $A(x) = -\int \frac{g(x)}{W} y_2(x)\,dx + C_1$ $\quad B(x) = \int \frac{g(w)}{W} (y_1)(x)\,dx + C_2$

$$y_p = -y_1(x) \int \frac{g(x)y_2(x)}{W(x)}\,dx + y_2(x) \int \frac{g(x)y_1(x)}{W(x)}\,dx$$

where $g(x) = \frac{r(x)}{a_0(x)}$ and $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Q Find the general solution of the equation

$$y'' + 3y' + 2y = 2e^x \quad \left\{a_0y'' + a_1y' + a_2y = r(x)\right\}$$

Sol^n SF $(D^2+3D+2)y = 2e^x$ $\qquad a_0 = 1$

AE $D^2+3D+2 = 0$ $\qquad a_1 = 3$ $\qquad r(x) = 2e^x$

$a_2 = 2$

$$(D+1)(D+2) = 0$$

$$\Rightarrow D = -1, -2$$

$$y_c(x) = C_1e^{-x} + C_2e^{-2x} \quad \text{--- (1)}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x}$$

$$= -e^{-3x}$$

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$$g(x) = \frac{r(x)}{a_0(x)}$$

$$= \frac{2e^x}{1} = 2e^x$$

$$y_p = -y_1 \int \frac{g(x)\, y_2(x)}{W(x)} + y_2 \int \frac{g(x)}{W(x)} \times y_1(x)\, dx$$

$$y_p = -e^{-x} \int \frac{2e^x \times e^{-2x}}{-e^{-3x}}\, dx + e^{-2x} \int \frac{2e^x}{-e^{-3x}} \times e^{-x}\, dx$$

$$= 2e^{-x} \int e^{2x}\, dx - 2e^{-2x} \int e^{3x}\, dx$$

$$= \cancel{2}e^{-x} \times \frac{e^{2x}}{\cancel{2}} - 2 \times e^{-2x} \times \frac{e^{3x}}{3}$$

$$\Rightarrow e^x - \frac{2}{3}e^x = \frac{1}{3}e^x$$

$$y(x) = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{3}e^x.$$

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Q $y''+16y = 32\sec 2x$ using the method of variation of parameter

Soln:- Given equation is

$$y''+16y = 32\sec 2x \quad \text{—} (1).$$

$a_0 = 1$

SF $\quad [D^2+16]y = 32\sec 2x$

AE $\quad D^2+16=0$

$\Rightarrow D = \pm 4i$

$$y_c(x) = C_1 \underbrace{\cos 4x}_{y_1} + C_2 \underbrace{\sin 4x}_{y_2}$$

Take, $\quad y_1 = \cos 4x$ and $y_2 = \sin 4x$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4\cos^2 4x + 4\sin^2 4x$$
$$= 4$$

$$g(x) = \frac{r(x)}{a_0(x)}$$

$$g(x) = \frac{32\sec 2x}{1} = 32\sec 2x$$

$$y_P = -y_1 \int \frac{g(x)y_2(x)}{W(x)} + y_2 \int \frac{g(x)y_1(x)}{W(x)}\,dx$$

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$$y_P = -\cos 4x \int \frac{32\sec 2x \times \sin 4x}{4} dx + \sin 4x \int \frac{(32\sec 2x)(\cos 4x)}{4} dx$$

$$y_P = -8\cos 4x \int \frac{2\sin 2x \cos 2x}{\cos 2x} dx + 8\sin 4x \int \frac{1}{\cos 2x}[2\cos^2 2x - 1] dx$$

$$y_P = -8\cos 4x \int 2\sin 2x \, dx + 8\sin 4x \int (2\cos 2x - \sec 2x) dx$$

$$y_P = -16\cos 4x \times \left[\frac{-1}{2}\cos 2x\right] + 8\sin 4x \left[\frac{2}{2}\sin 2x - \frac{\ln|\sec 2x + \tan 2x|}{2}\right]$$

$$y_P = 8\cos 2x \cos 4x + 8\sin 2x \sin 4x - 4\sin 2x \ln|\sec 2x + \tan 2x|$$

$$y_P = 8[\cos 2x] - 4\sin 2x \ln|\sec 2x + \tan 2x|$$

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Q find the general solution of the equation

$$y''' - 6y'' + 11y' - 6y = e^{-x}$$

SF $$[D^3 - 6D^2 + 11D - 6]y = e^{-x}$$

AE $$D = 1, 2, 3$$

$$y_c(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

we assume

$$y_1(x) = e^{x}, \quad y_2(x) = e^{2x}, \quad y_3 = e^{3x}$$

let

$$y_p(x) = A(x) y_1 + B(x) y_2 + C(x) y_3$$

$$A' y_1 + B' y_2 + C' y_3 = 0$$

$$A' y_1' + B' y_2' + C' y_3' = 0$$

$$A' y_1'' + B' y_2'' + C' y_3'' = g(x) \qquad \text{where } g(x) = \frac{r(x)}{a_0(x)}$$

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$$A'e^{x} + B'e^{2x} + C'e^{3x} = 0$$

$$A'e^{x} + 2B'e^{2x} + 3C'e^{3x} = 0$$

$$A'e^{x} + 4B'e^{2x} + 9C'e^{3x} = e^{-x}$$

$$W \text{ or } \Delta = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

$$\Delta_1 = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^{-x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{4x}$$

$$\Delta_2 = \begin{vmatrix} e^{x} & 0 & e^{3x} \\ e^{x} & 0 & 3e^{3x} \\ e^{x} & e^{-x} & 9e^{3x} \end{vmatrix} = -2e^{3x}$$

$$\Delta_3 = \begin{vmatrix} e^{x} & e^{2x} & 0 \\ e^{x} & 2e^{2x} & 0 \\ e^{x} & 4e^{2x} & e^{-x} \end{vmatrix} = e^{2x}$$

$$A' = \frac{\Delta_1}{\Delta} = \frac{e^{4x}}{2e^{6x}} \qquad B' = \frac{\Delta_2}{\Delta} = \frac{-2e^{3x}}{2e^{6x}}, \quad C' = \frac{\Delta_3}{\Delta} = \frac{e^{2x}}{2e^{6x}}$$

$$A' = \frac{1}{2}e^{-2x} \qquad B' = \frac{-1}{e^{3x}} \qquad C' = \frac{1}{2e^{4x}}$$

$$A = -\frac{1}{4}e^{-2x} + C_1 \qquad B = \frac{1}{3}e^{-3x} + C_2 \qquad C = -\frac{1}{8}e^{-4x} + C_3$$

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$$y(x) = Ay_1 + By_2 + Cy_3$$

$$= \left[\frac{-1}{4}e^{-2x} + c_1\right]y_1 + \left[\frac{1}{3}e^{-3x} + c_2\right]y_2 + \left[\frac{-1}{8}e^{-4x} + c_3\right]y_3$$

$$y(x) = -\frac{1}{4}e^{-2x}y_1 + c_1y_1 + \frac{1}{3}e^{-3x}y_2 + c_2y_2 - \frac{1}{8}e^{-4x}y_3 + c_3y_3$$

$$= c_1y_1 + c_2y_2 + c_3y_3 - \frac{1}{4}e^{-2x} \cdot e^{x} + \frac{1}{3}e^{-3x}e^{2x} - \frac{1}{8}e^{-4x}e^{3x}$$

$$= c_1e^{x} + c_2e^{2x} + c_3e^{3x} - \frac{1}{24}e^{-x}$$ Ans

Simultaneous linear Equation

Problem 1 Find the solution of the system of equations

$$\frac{dy_1}{dt} + \frac{2dy}{dt} - 2y_1 - y_2 = e^{2t}$$

$$\frac{dy_2}{dt} + y_1 - 2y_2 = 0$$

Method 1

$$y_1' + 2y_2' - 2y_1 - y_2 = e^{2t} \quad \text{---} (1)$$

$$y_2' + y_1 - 2y_2 = 0 \quad \text{---} (2)$$

from (2)

$$y_1 = 2y_2 - y_2' \quad \text{---} (3)$$

$$\Rightarrow \quad y_1' = 2y_2' - y_2'' \quad \text{---} (4)$$

Substitute these values in Eq (1)

$$(2y_2' - y_2'') + 2y_2' - 2(2y_2 - y_2') - y_2 = e^{2t}$$

or $$(2y_2' - y_2'') + 2y_2' - 4y_2 + 2y_2' - y_2 = e^{2t}$$

or $$-y_2'' + 6y_2' - 5y_2 = e^{2t}$$

or $$y_2'' - 6y_2' + 5y_2 = -e^{2t}$$

SF $$[D^2 - 6D + 5]\, y_2 = -e^{2t}$$

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<u>A6</u>

$$D^2-6D+5=0$$

$$D^2-5D-D+5=0$$

$$\Rightarrow D(D-5)-(D-5)=0$$

$$\Rightarrow (D-1)(D-5)=0$$

$$D=1,5$$

$$y_c = c_1e^{t}+c_2e^{5t}$$

$$y_P = -\frac{1}{D^2-6D+5}e^{2t}$$

$$= -\frac{1}{4-12+5}e^{2t} = -\frac{1}{-3}e^{2t} = \frac{1}{3}e^{2t} \quad \underline{\text{Ans}}$$

$$y_2 = c_1e^{t}+c_2e^{5t}+\frac{1}{3}e^{2t}$$

from (3)

$$y_1 = 2y_2 - y_2'$$

$$y_1 = 2\left[c_1e^{t}+c_2e^{5t}+\frac{1}{3}e^{2t}\right]-\left[c_1e^{t}+5c_2e^{5t}+\frac{2}{3}e^{2t}\right]$$

$$y_1 = c_1e^{t}-3c_2e^{5t}$$

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Method 2

$$\frac{dy_1}{dt} + \frac{2dy_2}{dt} - 2y_1 - y_2 = e^{2t}$$

$$\frac{dy_2}{dt} + y_1 - 2y_2 = 0$$

$$\left(\frac{d}{dt}(y_1) - 2y_1\right) + \left[2\frac{d}{dt}(y_2) - y_2\right] = e^{2t}$$

$$\Rightarrow \begin{cases} [D-2]y_1 + (2D-1)y_2 = e^{2t} \quad \text{--- (1)} \\ y_1 + (D-2)y_2 = 0 \quad \text{--- (2)} \end{cases}$$

Multiply Eq (2) by $(D-2)$

$$\cancel{(D-2)y_1} + (2D-1)y_2 = e^{2t}$$

$$- \quad \cancel{(D-2)y_1} + (D-2)^2 y_2 = 0$$

$$[(2D-1) - (D-2)^2]y_2 = e^{2t}$$

$$[2D-1-D^2-4+4D]y_2 = e^{2t}$$

or $[-D^2+6D-5]y_2 = e^{2t}$

or $[D^2-6D+5]y_2 = -e^{2t}$

AE $\quad D^2-6D+5=0$

$(D-1)(D-5)=0$

$D=1,5$

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$y_2^c = c_1 e^{t} + c_2 e^{5t}$

$y_2^p = -\frac{1}{D^2-6D+5} e^{2t} = \frac{-1}{4-12+5} e^{2t}$

$y_2 = c_1 e^{t} + c_2 e^{5t} + \frac{1}{3} e^{2t}$

from Eq (2).

$y_1 = (2-D) y_2$ $D = \frac{d}{dt}$

$y_1 = (2 y_2 - D y_2)$

$y_1 = [2[c_1 e^{t} + c_2 e^{5t} + \frac{1}{3} e^{2t}] - \frac{d}{dt}[c_1 e^{t} + c_2 e^{5t} + \frac{1}{3} e^{2t}]$

$y_1 = [c_1 e^{t} - 3c_2 e^{5t}]$. As

Method 3

$$(D-2)\,y_1 + (2D-1)\,y_2 = e^{2t}$$

$$y_1 + (D-2)\,y_2 = 0.$$

$$\Delta = \begin{vmatrix} D-2 & 2D-1 \\ 1 & D-2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} e^{2t} & 2D-1 \\ 0 & D-2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} D-2 & e^{2t} \\ 1 & 0 \end{vmatrix}$$

Δ

$$y_1 = \frac{\Delta_1}{\Delta} \qquad y_2 = \frac{\Delta_2}{\Delta}.$$

$$\Delta = (D-2)^2 - (2D-1) = D^2 - 6D + 5 \neq 0$$

$$\Delta_1 = (D-2)\,e^{2t}$$

$$\Delta_2 = -e^{2t}$$

$$\cancel{y_1 = \frac{(D-2)e^{2t}}{(D-2)(D-3)}}$$

$$y_1 = \frac{(D-2)e^{2t}}{D^2-6D+5} \Rightarrow (D^2-6D+5)\,y_1 = (D-2)\,e^{2t}$$

$$= 2e^{2t} - 2e^{2t}$$

$$= 0.$$

$\Rightarrow$

$$(D^2 - 6D + 5)\,y_1 = 0$$

$\because$ this is a homo. function

$$\therefore y(x) = c_1 e^{x} + c_2 e^{5x}$$

g.s = C.F

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Similarly find y_2

$$y_2 = \frac{\Delta_2}{\Delta} = \frac{-e^{2t}}{D^2-6D+5}$$

$\Rightarrow$ $[D^2-6D+5]y_2 = -e^{2t}$

$\Rightarrow$ find y_2^c and y_2^p.

~~and~~ and ~~find~~ $\boxed{y_2(t) = y_2^c + y_2^p}$.